Chapter 1

Time-Frequency Concepts [0]

Time-frequency signal analysis and processing (TFSAP) concerns the analysis and processing of signals with time-varying frequency content. Such signals are best represented by a **time-frequency distribution** (**TFD**), which is intended to show how the energy of the signal is distributed over the two-dimensional time-frequency space. Processing of the signal may then exploit the features produced by the concentration of signal energy in two dimensions (time *and* frequency) instead of only one (time *or* frequency).

The first chapter begins the introductory tutorial which constitutes Part I of the book. This tutorial updates the one given in [1] by including recent advances, refining terminology, and simplifying both the presentations of concepts and formulations of methodologies. Reading the three chapters of Part I will facilitate the understanding of the later chapters.

The three sections of Chapter 1 present the key concepts needed to formulate time-frequency methods. The first (Section 1.1) explains why time-frequency methods are preferred for a wide range of applications in which the signals have time-varying characteristics or multiple components. Section 1.2 provides the signal models and formulations needed to describe temporal and spectral characteristics of signals in the time-frequency domain. It defines such basic concepts as analytic signals, the Hilbert transform, the bandwidth-duration product and asymptotic signals. Section 1.3 defines the key quantities related to time-frequency methods, including the instantaneous frequency (IF), time-delay (TD) and group delay (GD).

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1.1 The Need for a Time-Frequency Distribution (TFD)

The two classical representations of a signal are the time-domain representation $s(t)$ and the frequency-domain representation $S(f)$. In both forms, the variables t and f are treated as mutually exclusive: to obtain a representation in terms of one variable, the other variable is "integrated out". Consequently, each classical representation of the signal is **non-localized** w.r.t. the excluded variable; that is, the frequency representation is essentially averaged over the values of the time representation at *all* times, and the time representation is essentially averaged over the values of the frequency representation at *all* frequencies.

In the time-frequency distribution, denoted by $\rho(t, f)$, the variables t and f are *not* mutually exclusive, but are present together. The TFD representation is **localized** in *t and f*.

1.1.1 Representation of Three Real-Life Signals

The usefulness of representing a signal as a function of both time *and* frequency is illustrated by considering three signals of practical importance:

1. **Sinusoidal FM signal:** Monophonic television sound, like monophonic FM radio, is transmitted on a frequency-modulated carrier. If the audio signal is a pure tone of frequency f_{m} (the modulating frequency), then the frequency of the carrier is of the form

$$f_i(t) = f_{\mathrm{c}} + f_{\mathrm{d}} \cos[2\pi f_{\mathrm{m}} t + \phi] \tag{1.1.1}$$

   where t is time, $f_i(t)$ is the frequency modulation law (FM law), f_{c} is the mean (or "center") carrier frequency, f_{d} is the peak frequency deviation, and ϕ allows for the phase of the modulating signal. The amplitude of the carrier is constant.

2. **Linear FM signal:** Consider a sinusoidal signal of total duration T, with constant amplitude, whose frequency increases from f_0 to $f_0 + B$ at a constant rate $\alpha = B/T$. If the origin of time is chosen so that the signal begins at $t = 0$, the FM law may be written

$$f_i(t) = f_0 + \alpha t \; ; \quad 0 \leq t \leq T \; . \tag{1.1.2}$$

   In an electronics laboratory, such a signal is called a **linear frequency sweep**, and might be used in an automated experiment to measure the frequency response of an amplifier or filter. In mineral exploration, a linear FM signal is called a **chirp** or **Vibroseis** signal, and is used as an acoustic "ping" for detecting underground formations [2, 3].

3. **Musical performance:** A musical note consists of a number of "components" of different frequencies, of which the lowest frequency is called the fundamental and the remainder are called overtones [4, p. 270]. These components are

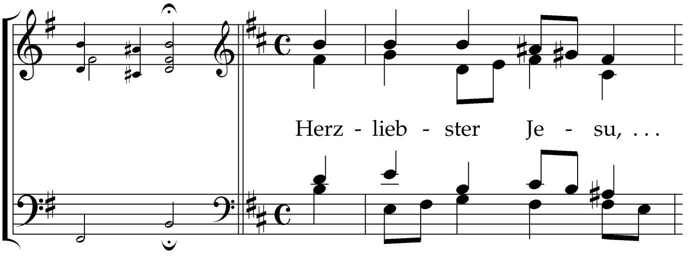

Fig. 1.1.1: An example of musical notation [5]. Roughly speaking, the horizontal dimension is time and the vertical dimension is frequency.

present during a specific time interval and may vary in amplitude during that interval. In modern musical notation, each note is represented by a "head". The vertical position of the head (together with other information such as the clef and key signature) indicates the *pitch*, i.e. the frequency of the most prominent component (usually the fundamental). The horizontal position of the head in relation to other symbols indicates the starting time, and the duration is specified by the shading of the head, attached bars, dots, stem and flags, and tempo markings such as *Allegro*. The choice of instrument—each instrument being characterized by its overtones and their relationships with the fundamental—is indicated by using a separate stave for each instrument or group of instruments, or a pair of staves for a keyboard instrument. Variations in amplitude are indicated by dynamic markings such as *mp* and *crescendo*. Fig. 1.1.1 illustrates the system. By scanning a set of staves vertically, one can see which fundamentals are present on which instruments at any given time. By scanning the staves horizontally, one can see the times at which a given fundamental is present on a given instrument.

Each of the three above signals has a time-varying frequency or time-varying "frequency content". Such signals are referred to as **non-stationary**.

The three examples described above are comprehensible partly because our sense of hearing readily interprets sounds in terms of variations of frequency or "frequency content" with time. However, conventional representations of a signal in the time domain or frequency domain do not facilitate such interpretation, as shown below.

1.1.2 Time-Domain Representation

Any signal can be described naturally as a function of time, which we may write $s(t)$. This representation leads immediately to the **instantaneous power**, given by $|s(t)|^2$, which shows how the energy of the signal is distributed over time; the total signal energy is

$$E = \int_{-\infty}^{\infty} |s(t)|^2 \, dt. \tag{1.1.3}$$

But the time-domain description has limitations, as may be seen by applying it to the above three examples:

1. The **sinusoidal FM signal** whose frequency satisfies Eq. (1.1.1) may be written
$$s_1(t) = A\cos\left(2\pi f_c t + \tfrac{f_d}{f_m}\sin[2\pi f_m t + \phi] + \psi\right) \tag{1.1.4}$$
where A is the amplitude and ψ a phase offset; the fraction f_d/f_m is called the **modulation index** and is equal to the peak phase deviation (in radians) from $2\pi f_c t$. This equation, by itself, does not clearly show how the frequency varies with time. If we imagine a graph of $s_1(t)$ vs. t, it would give the impression of an oscillating frequency, but determining the frequency as a function of time from the graph would be a tedious and imprecise exercise.

2. The **linear FM signal** whose frequency satisfies Eq. (1.1.2) may be written
$$s_2(t) = A\,\mathrm{rect}\left[\tfrac{t-T/2}{T}\right]\cos\left(2\pi[f_0 t + \tfrac{\alpha}{2}t^2] + \psi\right) \tag{1.1.5}$$
where, again, A is the amplitude and ψ a phase offset. The rect function is a rectangular pulse of unit height and unit duration, centered on the origin of time; that is,
$$\mathrm{rect}\,\tau = \begin{cases} 1 & \text{if } |\tau| \le 1/2\,; \\ 0 & \text{otherwise.} \end{cases} \tag{1.1.6}$$
Hence the rect[...] factor in Eq. (1.1.5) is equal to unity for $0 \le t \le T$, and zero elsewhere. But again it is not immediately apparent why Eq. (1.1.5) has the required FM law.

   The graph of $s_2(t)$ vs. t is shown on the left side of Fig. 1.1.2(a), for $\psi = 0$, $T = 64\,\mathrm{s}$, $f_0 = 0.1\,\mathrm{Hz}$ and $\alpha = (3/640)\,\mathrm{Hz\,s^{-1}}$. Although the graph gives a clear impression of a steadily increasing frequency, the exact FM law is not clearly and readily visible.

3. A **musical performance** can be represented as (for example) an air pressure curve at a particular point in space. Each such curve is a time-varying pressure, and may be converted by a microphone and amplifier into an electrical signal of the form $s_3(t)$. Indeed, music is routinely recorded and broadcast in this way. However, the function $s_3(t)$ is nothing like the form in which a composer would write music, or the form in which most musicians would prefer to read music for the purpose of performance. Neither is it of much use to a recording engineer who wants to remove noise and distortion from an old "vintage" recording. Musical waveforms are so complex that a graph of $s_3(t)$ vs. t would be almost useless to musicians and engineers alike.

These three examples show that the time-domain representation tends to obscure information about frequency, because it assumes that the two variables t and f are mutually exclusive.

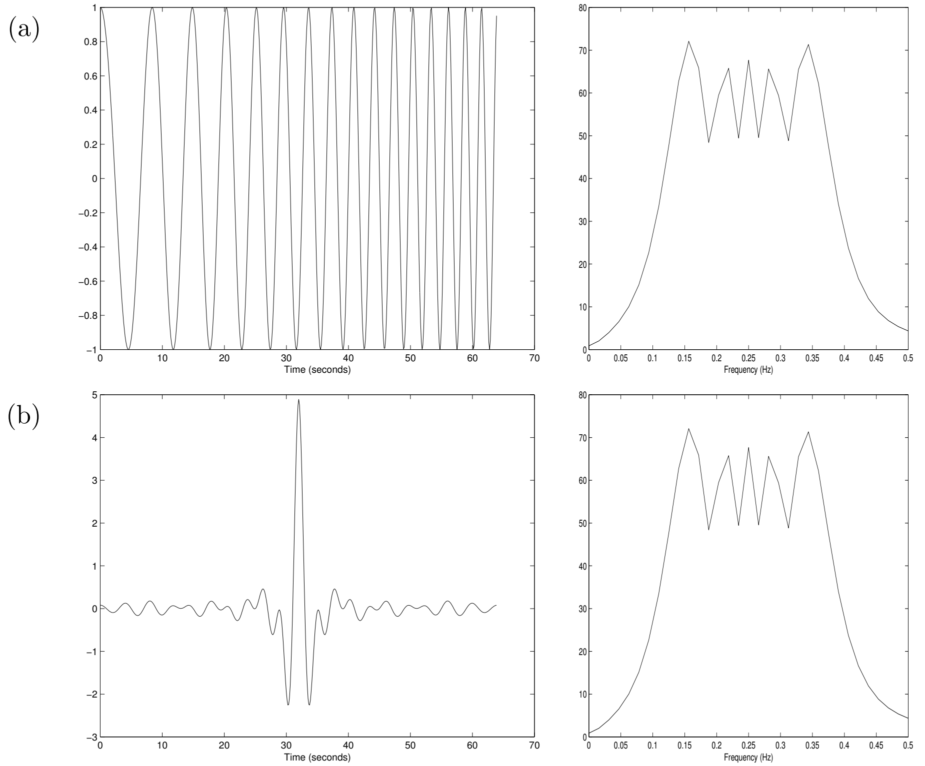

Fig. 1.1.2: The importance of phase: (a) time-domain representation (left) and magnitude spectrum (right) of a linear FM signal [Eq. (1.1.5)] with duration 64 seconds, starting frequency 0.1 Hz and finishing frequency 0.4 Hz; (b) time-domain representation and magnitude spectrum of another signal with the same magnitude spectrum as that in part (a). All plots use a sampling rate of 8 Hz.

1.1.3 Frequency-Domain Representation

Any practical signal $s(t)$ can be represented in the frequency domain by its Fourier transform $S(f)$, given by

$$S(f) = \underset{t \to f}{\mathcal{F}}\{s(t)\} \triangleq \int_{-\infty}^{\infty} s(t)\, e^{-j2\pi f t} dt. \tag{1.1.7}$$

For convenience, the relation between $s(t)$ and $S(f)$ may be written "$s(t) \underset{t \to f}{\longleftrightarrow} S(f)$" or simply "$s(t) \leftrightarrow S(f)$". The Fourier transform (FT) is in general complex; its magnitude is called the **magnitude spectrum** and its phase is called the **phase spectrum**. The square of the magnitude spectrum is the **energy spectrum** and shows how the energy of the signal is distributed over the frequency domain; the

total energy of the signal is

$$E = \int_{-\infty}^{\infty} |S(f)|^2 \, df = \int_{-\infty}^{\infty} S(f) \, S^*(f) \, df \tag{1.1.8}$$

where the superscripted asterisk (*) denotes the complex conjugate. Although the representation $S(f)$ is a function of frequency only—time having been "integrated out"—the FT is a complete representation of the signal because the signal can be recovered from the FT by taking the inverse Fourier transform (IFT):

$$s(t) = \underset{t \leftarrow f}{\mathcal{F}^{-1}} \{S(f)\} = \int_{-\infty}^{\infty} S(f) \, e^{j2\pi f t} df. \tag{1.1.9}$$

But the "completeness" of the FT representation does not make it convenient for all purposes, as may be seen by considering the same three examples:

1. **Sinusoidal FM signal:** If $\phi = \psi = 0$ in Eq. (1.1.4), the expression for $s_1(t)$ can be expanded into an infinite series as

$$s_1(t) = A \sum_{n=-\infty}^{\infty} J_n(\beta) \, \cos 2\pi (f_\mathrm{c} + n f_\mathrm{m}) t \tag{1.1.10}$$

   where β is the modulation index ($\beta = f_\mathrm{d}/f_\mathrm{m}$) and J_n denotes the Bessel function of the first kind, of order n [6, p. 226]. In the frequency domain, this becomes an infinite series of delta functions; one of these (the **carrier**) is at the mean frequency f_c , and the remainder (the **sideband** tones) are separated from the carrier by multiples of the modulating frequency f_m . Although the number of sideband tones is theoretically infinite, the *significant* ones[1] may be assumed to lie between the frequencies $f_\mathrm{c} \pm (f_\mathrm{d} + f_\mathrm{m})$ or, more conservatively, $f_\mathrm{c} \pm (f_\mathrm{d} + 2f_\mathrm{m})$. This information is essential if one is designing a tuning filter to isolate the TV audio carrier or separate one FM channel from adjacent channels. But it is inadequate if one is designing a modulator or demodulator, because its connection with the FM law is even more obscure than that of Eq. (1.1.4).

2. The magnitude spectrum of the **linear FM signal**: Eq. (1.1.5) is shown on the right side of Fig. 1.1.2(a), for $\psi = 0$, $T = 64\,\mathrm{s}$, $f_0 = 0.1\,\mathrm{Hz}$ and $\alpha = (3/640)\,\mathrm{Hz}/s$. The graph shows that magnitude is significant in the band corresponding to the frequency sweep range ($0.1\,\mathrm{Hz} < f < 0.4\,\mathrm{Hz}$), and furthermore that the energy is mostly confined to that band. However, the graph fails to show that the frequency is increasing with time. In other words, the

[1] For more detail, see Carlson [6], pp. 220–37. The above description considers only positive frequencies; similar comments apply to the negative frequencies. Theoretically, the lower sideband tones belonging to the positive-frequency carrier extend into the negative frequencies, while the corresponding sideband tones belonging to the negative-frequency carrier extend into the positive frequencies; but such "aliased" components are negligible if f_c and f_d are appropriately chosen.

magnitude spectrum tells us what frequencies are present in the signal, but not the "times of arrival" of those frequencies (the latter information, which we call **time delay**, is encoded in the phase spectrum).

The shortcomings of the magnitude spectrum may be seen even more clearly in Fig. 1.1.2(b), which shows a signal whose magnitude spectrum (right) is identical to that of the linear FM signal in Fig. 1.1.2(a), but whose appearance in the time domain (left) is very different from a linear FM.[2]

3. Similarly, the **musical performance** has a magnitude spectrum which tells us what frequencies are present, but not *when* they are present; the latter information is again encoded in the phase. The magnitude spectrum may exhibit as many as 120 peaks corresponding to the notes of the chromatic scale in the audible frequency range, and the relative heights of those peaks may tell us something about the tonality of the music (or whether it is tonal at all), but the timing of the notes will not be represented in the magnitude spectrum and will not be obvious from the phase spectrum.

These three examples show that the frequency-domain representation "hides" the information about timing, as $S(f)$ does not mention the variable t.

1.1.4 Joint Time-Frequency Representation

As the conventional representations in the time domain or frequency domain are inadequate in the situations described above, an obvious solution is to seek a representation of the signal as a *two-variable* function or distribution whose domain is the two-dimensional (t, f) space. Its constant-t cross-section should show the frequency or frequencies present at time t, and its constant-f cross-section should show the time or times at which frequency f is present. Such a representation is called a **time-frequency representation** (**TFR**) or **time-frequency distribution** (**TFD**).

As an illustration of what is desired, Fig. 1.1.3 shows one particular TFD of the linear FM signal of Eq. (1.1.5), for $\psi = 0$, $T = 64\,\mathrm{s}$, $f_0 = 0.1$ and $\alpha = (3/640)\,\mathrm{Hz\,s^{-1}}$. The TFD not only shows the start and stop times and the frequency range, but also clearly shows the variation in frequency with time. This variation may be described by a function $f_i(t)$, called the **instantaneous frequency** (**IF**). A signal may have more than one IF; for example, Fig. 1.1.4 shows a TFD of a sum of two linear FM signals, each of which has its own IF.[3] These IF features are not apparent in conventional signal representations.

Non-stationary signals for which a TFD representation may be useful occur not only in broadcasting, seismic exploration and audio, from which our three exam-

[2]The signal in part (b) of Fig. 1.1.2 was obtained from that in part (a) by taking the FFT, setting the phase to zero, taking the inverse FFT and shifting the result in time. It is *not* the product of a sinc function and a cosine function.

[3]N.B.: In Fig. 1.1.4 and in all subsequent graphs of TFDs, the labels on axes are similar to those in Fig. 1.1.3, but may be more difficult to read because of space constraints.

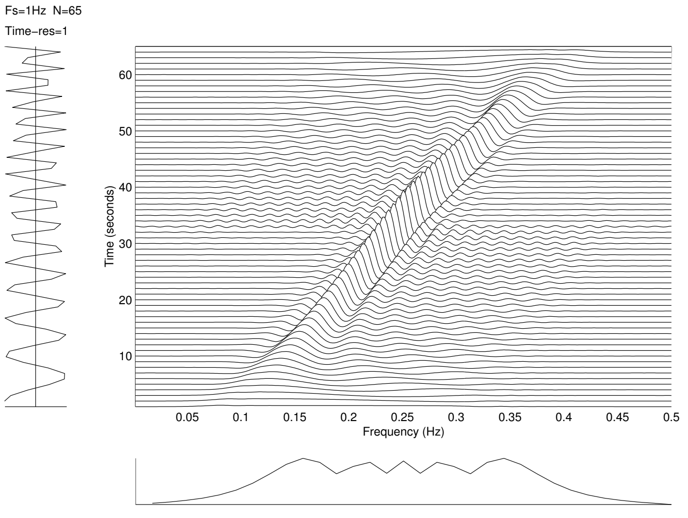

Fig. 1.1.3: A time-frequency representation of a linear FM signal [Eq. (1.1.5)] with duration 65 samples, starting frequency 0.1 and finishing frequency 0.4 (sampling rate 1 Hz). The time-domain representation appears on the left, and the magnitude spectrum at the bottom; this pattern is followed in all TFD graphs in Part I of this book.

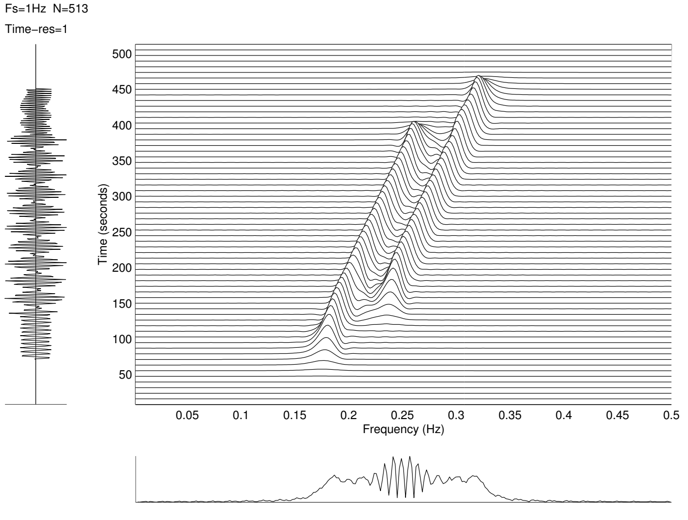

Fig. 1.1.4: A time-frequency representation of two linear FM signals with close parallel FM laws, duration 512 samples, frequency separation 0.04 (sampling rate 1 Hz).

ples are taken, but also in numerous other engineering and interdisciplinary fields such as telecommunications, radar, sonar, vibration analysis, speech processing and

medical diagnosis. **Time-frequency signal processing** (**TFSP**) is the processing of signals by means of TFDs.

1.1.5 Desirable Characteristics of a TFD

The use of a TFD for a particular purpose is inevitably based on particular assumptions concerning the properties of TFDs. To ensure that the conclusions of the analysis are sound, these assumptions must be identified and verified. Generalizing somewhat, we may say that the desirable properties of a TFD are those on which the most likely applications of TFDs depend. The following applications illustrate some likely uses of TFDs:

- Analyze the raw signal in the (t, f) domain so as to identify its characteristics such as time variation, frequency variation, number of components, relatives amplitudes, etc.
- Separate the components from each other and from the background noise by filtering in the (t, f) domain.
- Synthesize the filtered TFD in the time domain.
- Analyze specific components separately:
  - Track the instantaneous amplitude;
  - Track the instantaneous frequency;
  - Track the instantaneous bandwidth (spread of energy about the IF).
- Choose a mathematical model of the signal, showing clearly the significant characteristics, such as the IF.

These applications can be carried out using a TFD with the following properties:

1(a) The TFD is real (because energy is real).

1(b) The integral of the TFD over the entire (t, f) plane is the total energy of the signal (so that energy is conserved).

1(c) The integral over a rectangular region of the (t, f) plane, corresponding to a finite bandwidth and finite time interval, is approximately the energy of the signal in that bandwidth over that interval, provided that the bandwidth and interval are sufficiently large (see Section 1.2.5 for details).

2 For a monocomponent FM signal, the peaks of the constant-time cross-sections of the TFD should give the IF law which describes the signal FM law.

3 For a multicomponent FM signal, the dominant peaks of the TFD should reflect the components' respective FM laws; and the TFD should resolve any close components, as in Fig. 1.1.4.

A simple measure of the resolution of a TFD is the concentration of energy about the IF law(s) of a signal [Article 7.4 and Chapter 10].

The linear FM signal, because of its simple FM law, is a convenient test signal for verifying the first four properties. Property 3 may be tested by a sum of linear FM signals.

The above properties have further implications. For example, the resolution property (3) helps to ensure robustness in the presence of noise, while the last two properties (2 and 3) together allow discernment of the multiple IFs of multicomponent signals.

1.2 Signal Formulations and Characteristics in the (t, f) Domain

1.2.1 Signal Models used in (t, f) Methods

To represent signals such as the linear FM, several types of signal models are commonly used in TFSAP. The choice of model depends on the number and nature of the parameters needed to describe the signal. For example, a single sinusoid with constant frequency and normalized amplitude and phase is described by the equation

$$s_4(t) = \cos 2\pi f_{\mathrm{c}} t \ , \tag{1.2.1}$$

in which the only parameter is the frequency f_{c}. If the amplitude and phase are significant in the application, two more parameters are needed. A linear combination of such signals may be written in the form

$$s_5(t) = \sum_{k=1}^{M} a_k \cos(2\pi f_k t + \psi_k) \ , \tag{1.2.2}$$

which involves $3M$ parameters.

Because $s_4(t)$ and $s_5(t)$ contain terms (or "components") of constant amplitude, frequency and phase, they are completely and clearly described by Fourier transforms; no time dependence is required. But a sinusoidal FM signal or chirp signal requires a TFD. A musical audio signal also calls for a form of TFD (as is suggested by the notation in which music is written), and the TFD should clearly resolve the multiple components.

Further difficulties are raised by more complex signals such as

$$s_6(t) = \left(\sum_{k=1}^{M} a_k(t)\, e^{j2\pi \int_0^t f_k(\tau)\, d\tau} \right) + w(t) \tag{1.2.3}$$

where $a_k(t)$ is the *time-varying* amplitude of the k^{th} component, $f_k(t)$ is the *time-varying* frequency of the k^{th} component, and $w(t)$ is additive noise. The analysis of such a signal requires the ability not only to distinguish the *time-varying* components from each other in spite of their varying amplitudes and frequencies, but also

to separate them from the noise. Such comments still apply if the amplitude of the k^{th} component is a multiplicative noise factor $m_k(t)$, as in the signal

$$s_7(t) = \left(\sum_{k=1}^{M} m_k(t)\, e^{j2\pi \int_0^t f_k(\tau)\, d\tau} \right) + w(t). \tag{1.2.4}$$

In this case, an even more specialized analysis will be required.

1.2.2 Analytic Signals

It is well known that a signal $s(t)$ is real if and only if (iff)

$$S(-f) = S^*(f) \ , \tag{1.2.5}$$

where $S(f)$ is the Fourier transform of $s(t)$. In other words, a real signal is one that exhibits **Hermitian symmetry** between the positive-frequency and negative-frequency components, allowing the latter to be deduced from the former. Hence the negative-frequency components of a real signal *may be eliminated from the signal representation without losing information.* In the case of a real lowpass signal, removal of negative frequencies has two beneficial effects. First, it halves the total bandwidth, allowing the signal to be sampled at half the usual Nyquist rate without aliasing [7,8]. Second, it avoids the appearance of some interference terms generated by the interaction of positive and negative components in quadratic TFDs (to be treated in detail in Section 3.1.2).

Definition 1.2.1: *A signal $z(t)$ is said to be* **analytic** *iff*

$$Z(f) = 0 \quad \textit{for} \quad f < 0 \ , \tag{1.2.6}$$

where $Z(f)$ is the Fourier transform of $z(t)$.

In other words, an analytic signal contains *no negative frequencies*; it may have a spectral component at zero frequency (DC).

Theorem 1.2.1: *The signal*

$$z(t) = s(t) + jy(t) \ , \tag{1.2.7}$$

where $s(t)$ and $y(t)$ are real, is analytic with a real DC component, if and only if

$$Y(f) = (-j \operatorname{sgn} f)\, S(f) \tag{1.2.8}$$

where $S(f)$ and $Y(f)$ are the FTs of $s(t)$ and $y(t)$, respectively, and where

$$\operatorname{sgn} \xi \triangleq \begin{cases} -1 & \textit{if} \ \ \xi < 0 \ ; \\ 0 & \textit{if} \ \ \xi = 0 \ ; \\ +1 & \textit{if} \ \ \xi > 0 \ . \end{cases} \tag{1.2.9}$$

Proof: Take the FT of Eq. (1.2.7) and use Eq. (1.2.6). ■

1.2.3 Hilbert Transform; Analytic Associate

If the Fourier transforms of $s(t)$ and $y(t)$ are related according to Eq. (1.2.8), we say that $y(t)$ is the **Hilbert transform** of $s(t)$, and we write

$$y(t) = \mathcal{H}\{s(t)\}\,. \tag{1.2.10}$$

Hence we may restate Theorem 1.2.1 as follows: *A signal is analytic with a real DC component if and only if its imaginary part is the Hilbert transform of its real part.*

By invoking the "if" form of Theorem 1.2.1 and restating the sufficient condition in terms of Eq. (1.2.7), we may now see the practical significance of the theorem and the Hilbert transform: Given a real signal $s(t)$, we can construct the complex signal

$$z(t) = s(t) + j\mathcal{H}\{s(t)\} \tag{1.2.11}$$

and know that $z(t)$ is analytic. This $z(t)$ is called the analytic signal "corresponding to" or "associated with" the real signal $s(t)$. In this book, for convenience, we shall usually call $z(t)$ the **analytic associate** of $s(t)$.

By taking the IFT of Eq. (1.2.8) and applying Eq. (1.2.10), we arrive at the following concise definition of the Hilbert transform:

Definition 1.2.2: *The* **Hilbert transform** *of a signal $s(t)$, denoted by $\mathcal{H}\{s(t)\}$, is*

$$\mathcal{H}\{s(t)\} = \underset{t\leftarrow f}{\mathcal{F}^{-1}} \left\{ (-j\,\mathrm{sgn}\, f) \underset{t\to f}{\mathcal{F}} \{s(t)\} \right\}. \tag{1.2.12}$$

where $\mathcal{F}\{\ldots\}$ denotes the Fourier transform.

In other words, the Hilbert transform of $s(t)$ is evaluated as follows:

1. Take the Fourier transform $S(f)$ of $s(t)$;
2. Multiply $S(f)$ by $-j$ for positive f, by $+j$ for negative f, and by zero for $f = 0$;
3. Take the inverse Fourier transform.

According to step 2 of the above procedure, a Hilbert transformer introduces a phase lag of 90 degrees (as $-j = e^{-j\pi/2}$), producing a signal in **quadrature** to the input signal. The effect is well illustrated by the following result, which is easily verified using Definition 1.2.2 and a table of transforms:

Example 1.2.1: *If f_0 is a positive constant, then*

$$\mathcal{H}\{\cos(2\pi f_0 t)\} = \sin(2\pi f_0 t) \tag{1.2.13}$$

$$\mathcal{H}\{\sin(2\pi f_0 t)\} = -\cos(2\pi f_0 t). \tag{1.2.14}$$

It would be convenient if the pattern of Example 1.2.1 were applicable to modulated signals; so that for example, we could say

$$\mathcal{H}\{a(t) \cos \phi(t)\} = a(t) \sin \phi(t) \ , \tag{1.2.15}$$

which would imply that the analytic associate of the real signal $s(t) = a(t) \cos \phi(t)$ is

$$\begin{aligned} z(t) &= a(t) \cos \phi(t) + j\mathcal{H}\{a(t) \cos \phi(t)\} \\ &= a(t) \cos \phi(t) + j a(t) \sin \phi(t) \\ &= a(t)\, e^{j\phi(t)}. \end{aligned} \tag{1.2.16}$$

The condition under which Eq. (1.2.15) holds is that the variation of $a(t)$ is sufficiently slow to ensure "spectral disjointness", i.e. to avoid overlap between the spectrum of $a(t)$ and the spectrum of $\cos \phi(t)$.

Eq. (1.2.8) indicates that the transfer function of a Hilbert transformer is $-j \operatorname{sgn} f$. The corresponding impulse response is

$$\mathop{\mathcal{F}^{-1}}_{t \leftarrow f} \{-j \operatorname{sgn} f\} = \frac{1}{\pi t}. \tag{1.2.17}$$

Using this result and applying the convolution property to Eq. (1.2.12), we obtain a definition of the Hilbert transform in the time domain:

$$\mathcal{H}\{s(t)\} = s(t) * \frac{1}{\pi t} \tag{1.2.18}$$

$$= \frac{1}{\pi} \, \text{p.v.} \left\{ \int_{-\infty}^{\infty} \frac{s(\tau)}{t - \tau} \, d\tau \right\} \ , \tag{1.2.19}$$

where p.v.$\{\ldots\}$ denotes the Cauchy **principal value** of the improper integral [9], given in this case by[4]

$$\lim_{\delta \to 0} \left[\int_{-\infty}^{t-\delta} \frac{s(\tau)}{t - \tau} \, d\tau + \int_{t+\delta}^{\infty} \frac{s(\tau)}{t - \tau} \, d\tau \right]. \tag{1.2.20}$$

1.2.4 Duration, Bandwidth, BT Product

1.2.4.1 Finite Duration and Finite Bandwidth

In real life, signals are observed for finite periods of time and processed by devices with finite usable bandwidths. If a signal $s(t)$ has the Fourier transform $S(f)$, the **duration** of the signal is the smallest range of times outside which $s(t) = 0$, while the **bandwidth** of the signal is the smallest range of frequencies outside which $S(f) = 0$. These definitions, as we shall see, lead to the conclusion that a finite

[4] In practice Eq. (1.2.18) is rarely used, because the FT properties make it easier to work in the frequency domain.

duration implies infinite bandwidth and vice versa. In practice, however, any useful signal must have a beginning and an end (finite duration) and its FT must be within the frequency range of the measuring/processing equipment (finite bandwidth).[5] In practice, therefore, the strict definitions need to be relaxed in some way.

A **time-limited signal**, of **duration** T centered at time $t = 0$, can be expressed as

$$s_T(t) = s(t)\,\mathrm{rect}[t/T] \ , \tag{1.2.21}$$

where the subscript T indicates the duration [see Eq. (1.1.6)].

The FT of $s_T(t)$ is

$$S_T(f) = S(f) \underset{f}{*} T\,\mathrm{sinc}\, fT \tag{1.2.22}$$

where $\underset{f}{*}$ denotes convolution in frequency. Thus the bandwidth of $S_T(f)$ is infinite.

If, in order to avoid the effects of discontinuities, we replace $\mathrm{rect}[t/T]$ with a smoother window $w(t)$ of the same duration T, we can write

$$s_T(t) = s(t)\,\mathrm{rect}[t/T]\,w(t) \ , \tag{1.2.23}$$

whose Fourier transform still involves a convolution with $\mathrm{sinc}\, fT$, giving an infinite bandwidth.

In analogy to the time-limited case, a **band-limited signal**, of bandwidth B centered at the origin, can be expressed in the frequency domain as

$$S_B(f) = S(f)\,\mathrm{rect}[f/B] \tag{1.2.24}$$

In the time domain, the signal is given by

$$\mathrm{s}_B(t) = s(t) \underset{t}{*} B\,\mathrm{sinc}\, Bt \tag{1.2.25}$$

which has an infinite duration. Thus, under the "usual" definitions of duration and bandwidth, a finite bandwidth implies infinite duration.

1.2.4.2 Effective Bandwidth and Effective Duration

If there is no finite bandwidth containing *all* the energy of the signal, there may still be a finite bandwidth containing *most* of the energy. Hence, for example, a bandwidth containing 99% of the signal energy might be accepted as a useful measure of the signal bandwidth. If the nominated fraction of the signal energy were confined between the frequencies f_{min} and f_{max}, the bandwidth would be $B = f_{\mathrm{max}} - f_{\mathrm{min}}$.

A less arbitrary but more relaxed measure of bandwidth is the so-called **effective bandwidth** B_e, defined by

$$B_e{}^2 = \frac{1}{E_s} \int_{-\infty}^{\infty} f^2 |S(f)|^2 \, df \tag{1.2.26}$$

[5] In practice, all acquisition and measuring systems are low-pass.

where $S(f)$ is the FT of the signal, and E_s is the total energy of the signal, given by

$$E_s = \int_{-\infty}^{\infty} |S(f)|^2\, df. \tag{1.2.27}$$

B_e^2 is the **second moment** of $|S(f)|^2$ w.r.t. frequency, about the origin ($f = 0$). For brevity, we call B_e^2 the "second moment of the signal w.r.t. frequency". As an aid to remembering the definition, note that if f were a random variable and $|S(f)|^2$ were its probability density function (p.d.f.), then we would have $E_s = 1$, so that B_e^2 would be the variance of f if the mean of f were zero. Thus the effective bandwidth B_e is analogous to the standard deviation of f. When we consider how little of a typical probability distribution falls within one standard deviation, we realize that the effective bandwidth is only a mathematical construction and should *not* be used as an estimate of the bandwidth required for accurate measurement and processing of the signal.

The definition of duration, like that of bandwidth, can be relaxed so as to obtain both finite bandwidth and finite duration. For example, one could define the duration T where a nominated fraction of the signal energy occurs. The so-called **effective duration** T_e , defined by

$$T_e{}^2 = \frac{1}{E_s} \int_{-\infty}^{\infty} t^2 |s(t)|^2\, dt \ , \tag{1.2.28}$$

is the second moment of $|s(t)|^2$ w.r.t. time, about the origin ($t = 0$). For brevity, we refer to T_e^2 as "the second moment of the signal w.r.t. time." The definitions of effective duration and effective bandwidth were proposed in 1946 by Gabor [10].[6]

Slepian [11] has proposed definitions of duration and bandwidth based on the accuracy of the detecting and measuring apparatus. Let us define a "time truncation" of $s(t)$ as a signal $\hat{s}(t)$ that satisfies

$$\hat{s}(t) = \begin{cases} 0 & \text{if } \ t < t_1 \\ s(t) & \text{if } \ t_1 \le t \le t_2 \\ 0 & \text{if } \ t > t_2 \end{cases} \tag{1.2.29}$$

where $t_2 > t_1$, so that the duration of $\hat{s}(t)$ is $t_2 - t_1$. Let a "frequency truncation" be defined similarly in the frequency domain. Then the **Slepian duration** of $s(t)$ is the duration of the shortest time-truncation $\hat{s}(t)$ that the apparatus cannot distinguish from $s(t)$, while the **Slepian bandwidth** of $s(t)$ is the bandwidth of the most narrow-band frequency-truncation $\hat{S}(f)$ that the apparatus cannot distinguish from $S(f)$.

Given suitable definitions of B and T, the **bandwidth-duration product** BT is self-explanatory. The BT product also has a practical significance: because a

[6]Dennis Gabor (1900–1979), a Hungarian-born electrical engineer who settled in Britain, is best known for the invention of holography (1947–8), for which he was awarded the Nobel Prize for Physics in 1971.

signal of total bandwidth B can be reconstructed from samples at the sampling rate B, the total number of samples required to represent the signal—i.e. the number of **degrees of freedom** in the signal—is equal to BT. In other words, BT is a measure of the *information richness* of the signal. This implies that there may be little point in attempting to extract large amounts of information from a signal with a small BT product.

Example 1.2.2: *For the Gaussian signal*

$$s(t) = e^{-\alpha^2 t^2}, \tag{1.2.30}$$

the effective duration is

$$T_e = \frac{1}{2\alpha}. \tag{1.2.31}$$

The FT of $s(t)$ is

$$S(f) = \frac{\sqrt{\pi}}{\alpha}\, e^{-\pi^2 f^2/\alpha^2}, \tag{1.2.32}$$

so that the effective bandwidth is

$$B_e = \frac{\alpha}{2\pi}. \tag{1.2.33}$$

From Eqs. (1.2.31) and (1.2.33),

$$B_e T_e = \frac{1}{4\pi}. \tag{1.2.34}$$

It can be shown that the Gaussian signal is the only signal for which this equality holds, and that for all other signals $B_e T_e > \frac{1}{4\pi}$ [10].

1.2.5 Asymptotic Signals

For signals with significant information content, it is desired to know not only the overall bandwidth B, but also the distribution of energy through the bandwidth, e.g. the frequencies present, their relative amplitudes, and the times during which they are significant. Similarly we may want to know not only the overall duration T, but also the distribution of energy throughout the duration, e.g. the times during which the signal is present, the relative amplitudes at those times, and the significant frequencies present during those times. Such signals may be modeled by the class of asymptotic signals.

Definition 1.2.3: *A signal $s(t)$ is* **asymptotic** *iff it has the following properties:*

(a) The duration T, as defined by Slepian, is finite;

(b) The bandwidth B, as defined by Slepian, is finite;

(c) The product BT is large (e.g. greater than 10);

(d) The amplitude is bounded so that

$$\int_{-\infty}^{\infty} |s(t)|^2 dt \tag{1.2.35}$$

is finite.

Asymptotic signals allow useful approximations for deriving simple signal models (e.g. to express analytic signals).

1.2.6 Monocomponent vs. Multicomponent Signals

A **monocomponent** signal is described in the (t, f) domain by one single "ridge", corresponding to an elongated region of energy concentration. Furthermore, interpreting the crest of the "ridge" as a graph of IF vs. time, we require the IF of a monocomponent signal to be a single-valued function of time. Fig. 1.1.3 shows an example of a monocomponent signal.

Such a monocomponent signal has an analytic associate of the form

$$z(t) = a(t)\, e^{j\phi(t)} \ , \tag{1.2.36}$$

where $\phi(t)$ is differentiable, and $a(t)$ is real and positive (being the instantaneous amplitude). If $s(t)$ itself is real and asymptotic, it can be expressed as

$$s(t) = a(t) \cos \phi(t). \tag{1.2.37}$$

A **multicomponent** signal may be described as the sum of two or more monocomponent signals. Fig. 1.1.4 shows an example of a multicomponent signal (composed of two components).

1.3 Instantaneous Frequency and Time-Delay

1.3.1 Instantaneous Frequency (IF)

Definition 1.3.1: *The* **instantaneous frequency** *of a monocomponent signal is*

$$f_i(t) = \frac{1}{2\pi}\, \phi'(t) \tag{1.3.1}$$

where $\phi(t)$ is the instantaneous phase of the signal.

This formulation is justified below by considering first a constant-frequency signal, then a variable-frequency signal.

Consider the amplitude-modulated signal

$$x(t) = a(t) \cos(2\pi f_c t + \psi) \tag{1.3.2}$$

where f_c and ψ are constant. As t increases by the increment $1/f_c$, the argument of the cosine function increases by 2π and the signal passes through one cycle. So the

period of the signal is $1/f_{\mathrm{c}}$, and the frequency, being the reciprocal of the period, is f_{c}. The same signal can be written as

$$x(t) = a(t)\cos\phi(t) \tag{1.3.3}$$

where

$$\phi(t) = 2\pi f_{\mathrm{c}} t + \psi\ , \tag{1.3.4}$$

from which we obtain

$$f_{\mathrm{c}} = \frac{1}{2\pi}\,\phi'(t). \tag{1.3.5}$$

Although the left-hand side of this equation (the frequency) has been assumed constant, the right-hand side would be variable if $\phi(t)$ were a nonlinear function. So let us check whether this result can be extended to a *variable frequency.*

Consider a signal whose analytic associate is

$$z(t) = a(t)\,e^{j\phi(t)} \tag{1.3.6}$$

where $a(t)$ and $\phi(t)$ are real and $a(t)$ is positive; then $a(t)$ is called the **instantaneous amplitude** and $\phi(t)$ is called the **instantaneous phase**. Let $\phi(t)$ be evaluated at $t = t_1$ and $t = t_2$, where $t_2 > t_1$. By the mean value theorem of elementary calculus, if $\phi(t)$ is differentiable, there exists a time instant t between t_1 and t_2 such that

$$\phi(t_2) - \phi(t_1) = (t_2 - t_1)\,\phi'(t). \tag{1.3.7}$$

Let p_i be the period of one particular oscillation of $z(t)$, and let $f_i = 1/p_i$. If $t_2 = t_1 + p_i$, then $\phi(t_2) = \phi(t_1) + 2\pi$, so that Eq. (1.3.7) becomes

$$2\pi = p_i\phi'(t)\ ; \tag{1.3.8}$$

that is,

$$f_i = \frac{\phi'(t)}{2\pi}. \tag{1.3.9}$$

Now t is an instant during a cycle of oscillation and f_i is the frequency of that oscillation, suggesting that the right-hand side be defined as the **instantaneous frequency** (**IF**) at time t, as in Definition 1.3.1 above.

Comparing Eqs. (1.2.37) and (1.3.6), we see that

$$s(t) = \mathrm{Re}\{z(t)\}\,. \tag{1.3.10}$$

where $\mathrm{Re}\{\ldots\}$ denotes the real part. Now let us define

$$y(t) \stackrel{\triangle}{=} \mathrm{Im}\{z(t)\} = a(t)\sin\phi(t) \tag{1.3.11}$$

where $\mathrm{Im}\{\ldots\}$ denotes the imaginary part.

Using Eq. (1.3.1), we can easily confirm that the signals described by Eqs. (1.1.4) and (1.1.5) have the IFs given in Eqs. (1.1.1) and (1.1.2) respectively.

Definition 1.3.1 is strictly meaningful only for a monocomponent signal, while a multicomponent signal ought to have a separate IF for each component. In particular, note that the IF of the sum of two signals is *not* the sum of their two IFs.

The IF is a detailed description of the frequency characteristics of a signal. This contrasts with the notion of "average frequency" defined next.

Definition 1.3.2: *The* **average frequency** *of a signal is*

$$f_0 = \frac{\displaystyle\int_0^\infty f|S(f)|^2\,df}{\displaystyle\int_0^\infty |S(f)|^2\,df}. \tag{1.3.12}$$

where $S(f)$ is the Fourier transform of the signal.

In other words, f_0 is the **first moment** of $|S(f)|^2$ w.r.t. frequency; that is, we define the "average" frequency as if $|S(f)|^2$ were the probability density function of the frequency.

Notice that if $|S(f)|^2$ were replaced by a TFD, f_0 would become a function of time, suggesting that perhaps the first moment of a TFD w.r.t. frequency is a measure of IF. The conditions under which this is true will be explained in due course (and summarized in the table on p. 75). For the moment, we simply note that any reasonable time-frequency representation of a signal should contain information about the IF laws of the components. In particular, it would be most convenient if the *crests* of the ridges in the (t, f) domain represented the IF laws.

1.3.2 IF and Time Delay (TD)

The IF of a signal indicates the dominant frequency of the signal at a given time. Let us now seek a **dual** or "inverse" of the IF, indicating the dominant time when a given frequency occurs.

If $z(t)$ is an analytic signal with the Fourier transform

$$Z(f) = A\,\delta(f - f_i) \tag{1.3.13}$$

where A is in general complex, the dominant frequency is f_i.

Taking the inverse FT of $Z(f)$ gives

$$z(t) = Ae^{j2\pi f_i t}. \tag{1.3.14}$$

The instantaneous phase of $z(t)$, denoted by $\phi(t)$, is

$$\phi(t) = \arg z(t) = 2\pi f_i t + \arg A \tag{1.3.15}$$

so that

$$f_i = \frac{1}{2\pi}\,\phi'(t). \tag{1.3.16}$$

Although this result has been obtained for constant frequency, the right-hand of Eq. (1.3.16) is also valid for a variable-frequency signal, as explained earlier.

Now let us repeat the argument with the time and frequency variables interchanged. The signal

$$z(t) = a\,\delta(t - \tau_{\mathrm{d}}) \ , \tag{1.3.17}$$

where a is in general complex, is an impulse at time τ_{d}. If we ask what is the "delay" of this signal, the only sensible answer is τ_{d}. The FT of this signal is

$$Z(t) = a\, e^{-j2\pi f \tau_{\mathrm{d}}} \tag{1.3.18}$$

and its phase, denoted by $\theta(f)$, is

$$\theta(f) = \arg Z(f) = -2\pi f \tau_{\mathrm{d}} + \arg a \tag{1.3.19}$$

so that

$$\tau_{\mathrm{d}} = -\frac{1}{2\pi}\,\theta'(f). \tag{1.3.20}$$

Again, although this result has been obtained for a constant τ_{d} , the right-hand side of Eq. (1.3.20) is well defined even if $\theta'(f)$ varies with f.

Definition 1.3.3: *If $z(t)$ is an analytic signal with the Fourier transform $Z(f)$, then the* **time delay** *(TD) of $z(t)$, denoted by $\tau_{\mathrm{d}}(f)$, is*

$$\tau_{\mathrm{d}}(f) = -\frac{1}{2\pi}\,\theta'(f) \ , \tag{1.3.21}$$

where $\theta(f) = \arg Z(f)$

Notice that the definitions of τ_{d} and f_i are similar, except that time and frequency are interchanged and Eq. (1.3.21) has an extra minus sign; hence we say that time delay is the **dual** of instantaneous frequency.[7]

Seeing that the instantaneous frequency $f_i(t)$ is a function assigning a frequency to a given time, whereas the time delay $\tau_{\mathrm{d}}(f)$ is a function assigning a time to a given frequency, we may well ask whether the two functions are inverses of each other. Clearly they are not *always* inverses, because the IF function may not be invertible. So let us restrict the question to **invertible** signals, i.e. monocomponent signals whose IFs are monotonic functions of time.

One example of an invertible signal is the generalized Gaussian signal, i.e. a linear FM signal with a Gaussian envelope. Let such a signal peak at time t_0, with peak amplitude A, center frequency f_{c} , sweep rate α and decay constant β, and suppose

[7] The term "time delay" is well established, but tautological. The term "instantaneous frequency" is a quantity with the dimensions of frequency modified by an adjective indicating localization in time. The dual of this term should therefore be a quantity with the dimensions of time modified by an adjective indicating localization in frequency, e.g. "frequency-localized delay" or, more succinctly, "frequency delay".

that the instantaneous frequency is positive while the envelope is significant. Then the analytic associate of the signal is

$$z(t) = A \exp\left(j2\pi \left[f_{\mathrm{c}}[t-t_0] + \tfrac{\alpha+j\beta}{2}[t-t_0]^2\right]\right) \tag{1.3.22}$$

and using Eq. (1.3.16) with $a(t) = A\exp\left(-\pi\beta(t-t_0)^2\right)$, its IF is

$$f_i(t) = f_{\mathrm{c}} + \alpha[t-t_0]. \tag{1.3.23}$$

To find the inverse function of $f_i(t)$, we simply solve for t, obtaining

$$t = t_0 + \frac{f_i(t) - f_{\mathrm{c}}}{\alpha}\ , \tag{1.3.24}$$

which suggests that the time delay $\tau_{\mathrm{d}}(f)$ of $z(t)$ can be estimated by

$$\hat{\tau_{\mathrm{d}}}(f) = t_0 + \frac{f - f_{\mathrm{c}}}{\alpha}. \tag{1.3.25}$$

In general τ_{d} and $\hat{\tau_{\mathrm{d}}}$ are not equal, but converge as the BT product increases. As an example, Fig. 1.3.1 shows the IF (solid line) and TD (dotted line) of the generalized Gaussian signal $z(t)$ for two values of B_eT_e, where the subscript e means "effective". Note that the two curves are closer for the larger B_eT_e.

1.3.3 Mean IF and Group Delay (GD)

Let $z(t)$ be a bandpass analytic signal with center frequency f_{c}. Let its Fourier transform be

$$Z(f) = M(f - f_{\mathrm{c}})\, e^{j\theta(f)} \tag{1.3.26}$$

where the magnitude $M(f-f_{\mathrm{c}})$ and phase $\theta(f)$ are real. If the signal has **linear phase** in the support of $Z(f)$, i.e. if $\theta(f)$ is a linear function of f wherever $Z(f)$ is non-zero, we can let

$$\theta(f) = -2\pi\tau_{\mathrm{p}} f_{\mathrm{c}} - 2\pi\tau_{\mathrm{g}}[f - f_{\mathrm{c}}] \tag{1.3.27}$$

where τ_{p} and τ_{g} are real constants with the dimensions of time. Eq. (1.3.26) then becomes

$$Z(f) = M(f-f_{\mathrm{c}})\, e^{-j(2\pi\tau_{\mathrm{p}}f_{\mathrm{c}} + 2\pi\tau_{\mathrm{g}}[f-f_{\mathrm{c}}])} \tag{1.3.28}$$

$$= e^{-j2\pi f_{\mathrm{c}}\tau_{\mathrm{p}}}\, M(f-f_{\mathrm{c}})\, e^{-j2\pi\tau_{\mathrm{g}}[f-f_{\mathrm{c}}]}. \tag{1.3.29}$$

Taking the IFT of $Z(f)$, we find

$$z(t) = m(t-\tau_{\mathrm{g}})\, e^{j2\pi f_{\mathrm{c}}[t-\tau_{\mathrm{p}}]} \tag{1.3.30}$$

where $m(t)$ is the IFT of $M(f)$. Now because $M(f)$ is real, $m(t)$ is Hermitian [i.e. $m(-t) = m^*(t)$], so that $|m(t)|$ is even. Hence τ_{g} is the time about which the envelope function is symmetrical; for this reason, τ_{g} is called the **group delay** [12, pp. 123–124]. The phase of the oscillatory factor is $-2\pi f_{\mathrm{c}}\tau_{\mathrm{p}}$, wherefore τ_{p} is called the **phase delay**. These observations together with Eq. (1.3.27) lead to the following definition:

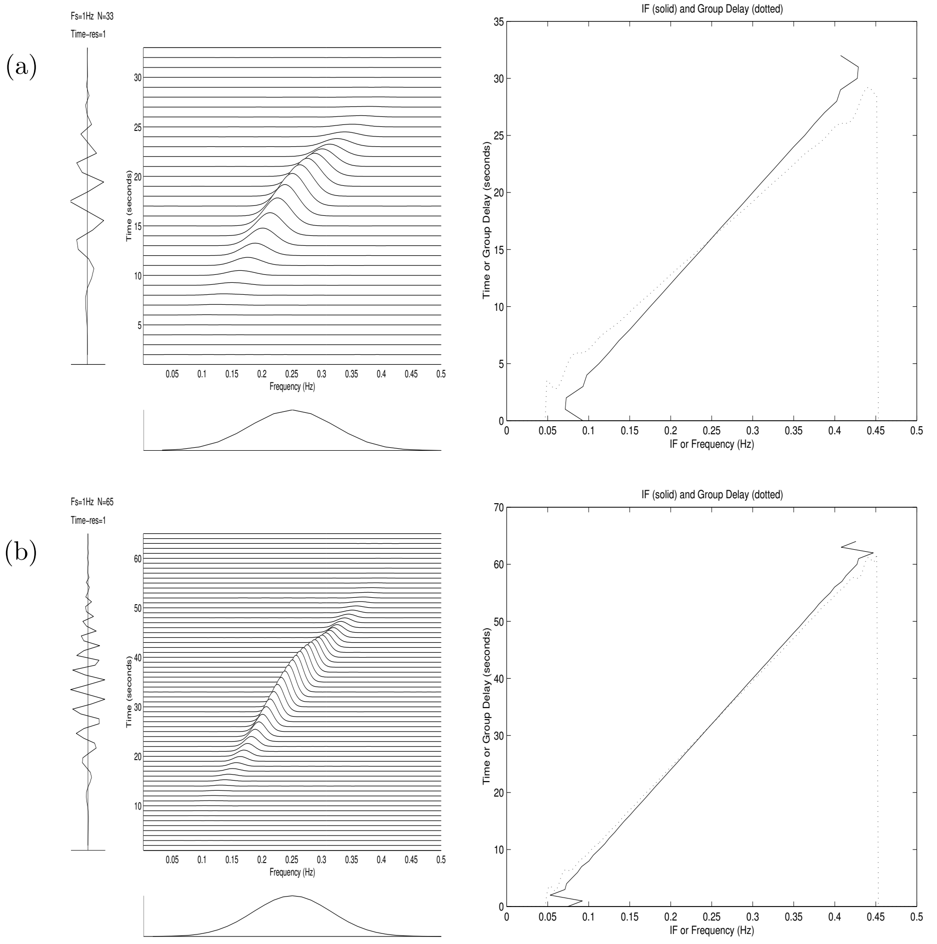

Fig. 1.3.1: Instantaneous frequency and time delay for a linear FM signal with a Gaussian envelope: (a) total duration $T = 33$ secs, $B_e T_e = 0.1806$; (b) $T = 65$ secs, $B_e T_e = 0.3338$. For each signal, the left-hand graph shows the time trace, spectrum and TFD [Wigner-Ville distribution, defined later in Eq. (2.1.17)], while the right-hand graph shows the IF (solid line) and time delay (dotted line). The vertical dotted segments are caused by truncation of the frequency range (to avoid finite-precision effects).

Definition 1.3.4: *If an analytic signal has the Fourier transform*

$$Z(f) = |Z(f)|\, e^{j\theta(f)} \,, \tag{1.3.31}$$

then the **group delay** *(GD) of the signal is*

$$\tau_{\mathrm{g}}(f) = -\frac{1}{2\pi}\, \theta'(f) \tag{1.3.32}$$

and the **phase delay** *of the signal is*

$$\tau_{\mathrm{p}}(f) = -\frac{1}{2\pi f}\,\theta(f). \tag{1.3.33}$$

Eq. (1.3.32) is found by differentiating Eq. (1.3.27) w.r.t. f, and Eq. (1.3.33) is found by putting $f_{\mathrm{c}} = f$ in Eq. (1.3.27). Whereas Eq. (1.3.27) assumes linear phase, the above definition is meaningful whether the phase is linear or not.

Further, Eq. (1.3.32) is the same as Eq. (1.3.21). But the physical interpretations are different; the time delay applies to an impulse, whereas the group delay applies to the envelope of a narrowband signal.

Now consider the dual of the above argument. Let $z(t)$ be a time-limited signal centered on $t = t_{\mathrm{c}}$, and let

$$z(t) = a(t - t_{\mathrm{c}})\, e^{j\phi(t)} \tag{1.3.34}$$

where $a(t)$ and $\phi(t)$ are real. If the signal has **constant instantaneous frequency** in the support of $z(t)$, i.e. if $\phi(t)$ is a linear function of t wherever $z(t)$ is non-zero, we can let

$$\phi(t) = 2\pi f_0 t_{\mathrm{c}} + 2\pi f_m[t - t_{\mathrm{c}}] \tag{1.3.35}$$

where f_0 and f_m are real constants with the dimensions of frequency. Eq. (1.3.34) then becomes

$$\begin{aligned} z(t) &= a(t - t_{\mathrm{c}})\, e^{j(2\pi f_0 t_{\mathrm{c}} + 2\pi f_m[t - t_{\mathrm{c}}])} && (1.3.36)\\ &= e^{j2\pi f_0 t_{\mathrm{c}}}\, a(t - t_{\mathrm{c}})\, e^{j2\pi f_m[t - t_{\mathrm{c}}]}. && (1.3.37) \end{aligned}$$

Taking the FT of $z(t)$, we find

$$Z(f) = A(f - f_m)\, e^{-j2\pi[f - f_0]t_{\mathrm{c}}} \tag{1.3.38}$$

where $A(f)$ is the FT of $a(t)$. Now because $a(t)$ is real, $A(f)$ is Hermitian, so that $|A(f)|$ is even. Hence f_m is the frequency about which the amplitude spectrum is symmetrical; for this reason, f_m is called the **mean IF**. Differentiating Eq. (1.3.35) w.r.t. t leads to the following definition:

Definition 1.3.5: *For the signal*

$$z(t) = |z(t)|\, e^{j\phi(t)}\ , \tag{1.3.39}$$

the **mean IF** *is*

$$f_m(t) = \frac{1}{2\pi}\,\phi'(t). \tag{1.3.40}$$

Thus the mean IF is the same as the IF defined earlier [Eq. (1.3.1)], but the physical interpretations are different. The IF has been derived for a tone (and earlier for a modulated sinusoid), whereas the mean IF applies to the spectrum of a short-duration signal.

1.3.4 Relaxation Time, Dynamic Bandwidth

For a linear FM signal, the instantaneous phase $\phi(t)$ is quadratic. So $\phi(t)$ can be expanded in a Taylor series about $t = t_0$:

$$\phi(t) = \phi(t_0) + \phi'(t_0)[t-t_0] + \tfrac{1}{2}\phi''(t_0)[t-t_0]^2 \tag{1.3.41}$$
$$= \phi(t_0) + 2\pi f_i(t_0)[t-t_0] + \tfrac{1}{2}2\pi f_i{}'(t_0)[t-t_0]^2. \tag{1.3.42}$$

The **relaxation time** T_r , as defined by Rihaczek [13, p. 374], is the duration over which the instantaneous phase deviates no more than $\pi/4$ from linearity. That is,

$$\left|\tfrac{1}{2}2\pi f_i{}'(t_0)T_\mathrm{r}^2/4\right| = \pi/4. \tag{1.3.43}$$

Solving this equation leads to the following definition:

Definition 1.3.6: *The* **relaxation time** *of a signal is*

$$T_\mathrm{r}(t) = \left|\frac{df_i(t)}{dt}\right|^{-1/2} \tag{1.3.44}$$

where $f_i(t)$ is the instantaneous frequency.

The dual of relaxation time, known as **dynamic bandwidth**, is the bandwidth over which the phase spectrum, assumed to be a quadratic function of frequency, deviates no more than $\pi/4$ from linearity. The result is as follows:

Definition 1.3.7: *The* **dynamic bandwidth** *of a signal is*

$$B_\mathrm{d}(f) = \left|\frac{d\tau_\mathrm{d}(f)}{df}\right|^{-1/2} \tag{1.3.45}$$

where $\tau_\mathrm{d}(f)$ is the time delay.

As the relaxation time is a measure of the time needed to observe significant variations in IF, so the dynamic bandwidth is a measure of the bandwidth needed to observe significant variations in time delay.

1.4 Summary and Discussion

Clear rationales are developed for justifying the need to use joint time-frequency representations for non-stationary signals such as FM signals.

An understanding of the concept of instantaneous frequency and its dual, time delay, is necessary for the interpretation of TFDs. The use of the analytic associate of a given real signal, rather than the signal itself, is useful for reducing the required sampling rate, and essential for obtaining an unambiguous instantaneous frequency.

The analytic associate is obtained via the Hilbert transform. The notions of BT product and monocomponent signals are introduced. For monocomponent asymptotic signals, analytic associates can be written using an exponential form. The next chapter introduces various formulations of time-frequency distributions and demonstrates the importance of using the analytic associate.

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